Bimonotone Subdivisions in High Dimensions

Haneul Shin
Mentor: Dr. Elina Robeva

Bergen County Academies

May 19, 2018
MIT PRIMES Conference

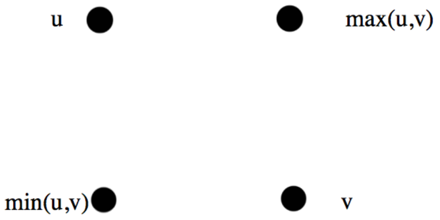
Supermodular Functions

Supermodular Functions

A function f from $\mathbb{R}^d$ to $\mathbb{R}$ is **supermodular** if for every u and v in $\mathbb{R}^d$, $f(u) + f(v) \leq f(\min(u, v)) + f(\max(u, v))$, where $\min(u, v)$ and $\max(u, v)$ are the coordinate-wise minimum and maximum of u and v, respectively.

- Example:

$$f(u) = 1, \;\; f(v) = 3, \;\; f(\min(u, v)) = 2, \;\; f(\max(u, v)) = 4$$



Main Goal

Characterize piecewise-linear concave supermodular functions. Application: Statistics

- Given random variables $X_1, ..., X_n \in \mathbb{R}$
- Distribution of random vector $(X_1, ..., X_n)$: density function $p : \mathbb{R}^n \to \mathbb{R}$
- If $p(x) = exp(f(x))$, where f is supermodular, then the random variables $X_1, ..., X_n$ are positively dependent.

Subdivisions in $\mathbb{R}^2$

Point Configuration in $\mathbb{R}^2$

A **point configuration** is a finite collection of points $A = \{a_1, ..., a_n\}$ in Euclidean space $\mathbb{R}^2$.

Subdivisions in $\mathbb{R}^2$

Point Configuration in $\mathbb{R}^2$

A **point configuration** is a finite collection of points $A = \{a_1, ..., a_n\}$ in Euclidean space $\mathbb{R}^2$.

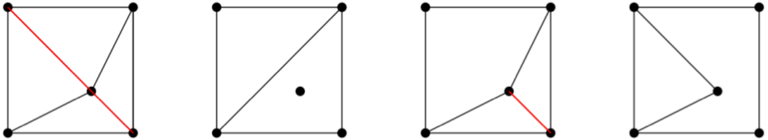
Subdivision in $\mathbb{R}^2$

A **subdivision** of a point configuration $A \subset \mathbb{R}^2$ is a collection S of convex polygons, all of whose vertices are points in A, that satisfies the following conditions.

1. The union of all of these polygons is the convex hull of A, denoted conv(A).
2. Any pair of these polygons either do not intersect, or they intersect in a common vertex, or in a common side.

- If all of the polygons in S are triangles, then S is a triangulation.

Subdivisions in $\mathbb{R}^2$: Example



(a)	(b)	(c)	(d)
Subdivision Triangulation	Subdivision Triangulation	Subdivision Not triangulation	Not subdivision Not triangulation

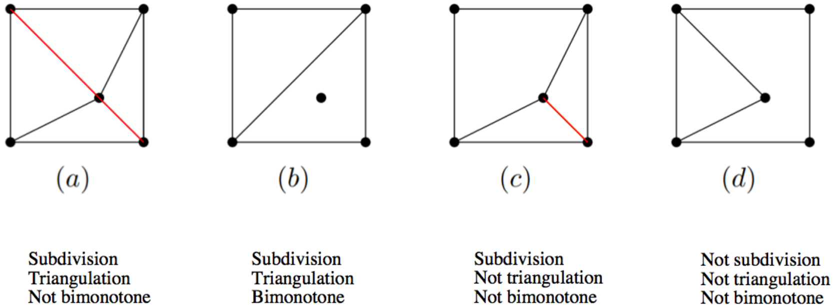
Bimonotone Subdivisions in $\mathbb{R}^2$

A subdivision S of a point configuration A is **bimonotone** if each of the polygons $P \in S$ is bimonotone. A polygon is bimonotone if each of its sides lies on a line given by an equality

$$ax + by + c = 0,$$

where $ab \leq 0$.

Bimonotone Subdivisions in $\mathbb{R}^2$: Example



Bimonotone Subdivisions in $\mathbb{R}^d$

- A point configuration is a finite collection of points in $\mathbb{R}^d$.
- A subdivision of a point configuration $A \subset \mathbb{R}^d$ is a collection S of convex polytopes that satisfy the conditions previously specified.

Bimonotone Subdivisions in $\mathbb{R}^d$

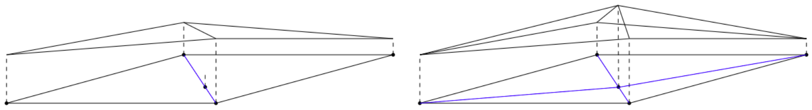
A subdivision $\mathcal{T}$ on a point configuration $A \subset \mathbb{R}^d$ is **bimonotone** if each of its polytopes $P \in T$ is bimonotone, or in other words, if each of its sides lies on a hyperplane defined by an equation

$$a_1x_1 + a_2x_2 + \cdots + a_dx_d + b = 0,$$

where all but at most two of the coefficients $a_1, ..., a_d$ are zero. If two of them are nonzero, say a_i and a_j, then $a_ia_j < 0$, i.e. they have opposite signs.

Driving Question

- Place a pole of some height y_i at each of the points in point configuration A
- Tent function: spread a piece of tarp on top over all of the poles
- This creates a subdivision of A
- The tent function $h_{X,y}$ is supermodular if and only if the subdivision it induces is bimonotone.



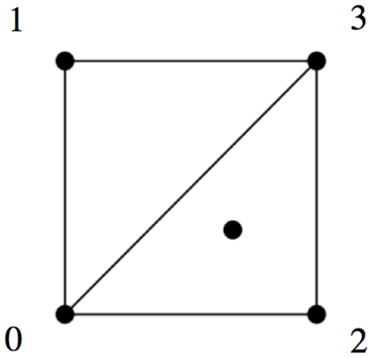
Driving Question: What is the characterization of tent-pole heights that give rise to a bimonotone subdivision?

TOPCOM

```
hshin1030@Hannah:/mnt/c/Windows/System32$ cd /mnt/c/topcom-0.17.8/src/
hshin1030@Hannah:/mnt/c/topcom-0.17.8/src$ ./chiro2alltriangs < /mnt/c/topcom-0.17.8/examples/r12.ch
iro
Evaluating Commandline Options ...
... done.
{{0,1,2,3},{0,1,2,4},{0,1,3,4},{0,2,3,5},{0,2,4,6},{0,2,5,6},{0,3,4,6},{0,3,5,6},{1,2,3,4},{2,3,4,7}
,{2,3,5,7},{2,4,6,10},{2,4,7,8},{2,4,8,9},{2,5,6,10},{2,5,7,8},{2,5,8,9},{3,4,6,10},{3,4,7,8},{3,4,8
,9},{3,5,6,10},{3,5,7,8},{3,5,8,9}}
{{0,1,2,3},{0,1,2,4},{0,1,3,4},{0,2,3,5},{0,2,4,6},{0,2,5,6},{0,3,4,6},{0,3,5,6},{1,2,3,4},{2,3,4,7}
,{2,3,5,7},{2,4,6,10},{2,4,7,8},{2,4,8,9},{2,5,6,10},{2,5,7,8},{2,5,8,9},{3,4,6,10},{3,4,7,8},{3,4,8
,9},{3,5,6,10},{3,5,7,9},{3,7,8,9},{5,7,8,9}}
{{0,1,2,3},{0,1,2,4},{0,1,3,4},{0,2,3,5},{0,2,4,6},{0,2,5,6},{0,3,4,6},{0,3,5,6},{1,2,3,4},{2,3,4,7}
,{2,3,5,7},{2,4,6,10},{2,4,7,8},{2,4,8,9},{2,5,6,10},{2,5,7,8},{2,5,8,9},{3,4,6,10},{3,4,7,8},{3,4,8
,9},{3,5,6,11},{3,5,7,8},{3,5,8,9},{3,5,10,11},{3,6,10,11},{5,6,10,11}}
{{0,1,2,3},{0,1,2,4},{0,1,3,4},{0,2,3,5},{0,2,4,6},{0,2,5,6},{0,3,4,6},{0,3,5,6},{1,2,3,4},{2,3,4,7}
,{2,3,5,7},{2,4,6,10},{2,4,7,8},{2,4,8,9},{2,5,6,10},{2,5,7,8},{2,5,8,9},{3,4,6,10},{3,4,7,8},{3,4,8
,9},{3,5,6,11},{3,5,7,9},{3,5,10,11},{3,6,10,11},{3,7,8,9},{5,6,10,11},{5,7,8,9}}
{{0,1,2,3},{0,1,2,4},{0,1,3,4},{0,2,3,5},{0,2,4,6},{0,2,5,6},{0,3,4,6},{0,3,5,6},{1,2,3,4},{2,3,4,7}
,{2,3,5,7},{2,4,6,10},{2,4,7,8},{2,4,8,9},{2,5,6,10},{2,5,7,8},{2,5,8,9},{3,4,6,10},{3,4,7,9},{3,5,6
,10},{3,5,7,9},{4,7,8,9},{5,7,8,9}}
```

Algorithm

- Input: Subdivision
- For each polytope, iterate over each of its faces and check if it is bimonotone
- Output: If the subdivision is bimonotone



Acknowledgements

I would like to sincerely thank

- My mentor, Dr. Elina Robeva for her invaluable support.
- Dr. Tanya Khovanova for her continued support.
- The MIT Math Department and the MIT-PRIMES program for this research opportunity.

Bibliography

R. Thomas. Lectures in Geometric Combinatorics 5: 37–45, (1991).

M. Cule, R. Samworth, and M. Stewart. Maximum likelihood estimation of a multi-dimensional log-concave density. J. R. Stat. Soc. Set. B Stat. Methodol., 72:545-607, (2010)

J. De Loera, J. Rambau, and F. Santos. Triangulations: structures for algorithms and applications. Springer-Verlag Berlin Heidelberg (2010)

E. Robeva, B. Sturmfels, N. Tran, and C. Uhler. Maximum likelihood estimation for totally positive densities. In preparation